Homework Set 3

1. (a) Prove that if X is affine over S then every S-morphism $\mathbb{P}^n_S \to X$ factors as $s \circ p$, where s is an S-morphism from S to X (that is, a section of $X \to S$), and $p\colon \mathbb{P}^n_S \to S$ is the structure morphism for $\mathbb{P}^n_S$ as a scheme over S. In particular, if $S = \operatorname{Spec}(k)$, where k is a field, then every k-morphism from $\mathbb{P}^n_k$ to an affine k-scheme X is constant, *i.e.*, it factors through the reduced one-point scheme $\operatorname{Spec}(k)$.

(b) Deduce that if k is a commutative ring (not the zero ring) and $n > 0$, then a vector bundle over $\mathbb{P}^n_k$ cannot be an affine scheme.

(c) Construct an example of an affine variety X and a morphism $\pi\colon X \to \mathbb{P}^n_k$ for some $n > 0$ such that X is an affine line bundle over $\mathbb{P}^n_k$, *i.e.*, $\mathbb{P}^n_k$ can be covered by open sets U such that $\pi^{-1}(U)$ is isomorphic to $\mathbb{A}^1_U$ as a scheme over U. Why does this not contradict part (b)?

[The hint I orginally gave on (c), to construct X as a surface $V(f)$ in $SL_2(k)$, and map it to $\mathbb{P}^1_k$ by acting with $SL_2(k)$ on a point $p \in \mathbb{P}^1_k$, was not helpful.

A better hint is to show that $\mathbb{P}^1 \times \mathbb{P}^1$ minus the diagonal is an affine variety X. In fact, X is then the quotient $SL_2(k)/T$, where T is the subgroup of diagonal matrices in $SL_2(k)$, and the projections to $\mathbb{P}^1$ coincide with the action of $SL_2(k)$ on either of the two points $p \in \mathbb{P}^1$ that are fixed by T. It is also true that X is isomorphic to a surface in $SL_2(k)$, but it does not embed in $SL_2(k)$ as a section of $SL_2(k) \to SL_2(k)/T = X$.]

2. Let X be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on X which is locally finitely generated. Let $S = S(\mathcal{F})$ be the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_X$, and let $V = \operatorname{Spec}(S)$, a scheme affine over X. (In the case where $\mathcal{F}$ is locally free, V is the geometric vector bundle whose sheaf of sections is dual to $\mathcal{F}$.) Note that V has a distinguished 'zero section' over X, corresponding to the surjective homomorphism $S \to S/S\mathcal{F} \cong \mathcal{O}_X$.

(a) Let $U \subseteq V$ be the open subset complementary to the zero section. Show that the image of $U \to X$ is equal to the support of $\mathcal{F}$; in particular it is closed.

(b) Find an example showing that if $\mathcal{F}$ is not assumed to be locally finitely generated, then the image of U is not necessarily closed, and is not necessarily equal to the support of $\mathcal{F}$. Such an example will also show that the image of $\operatorname{Proj}(S) \to X$ need not be closed when S is a quasi-coherent graded $\mathcal{O}_X$ algebra that is not locally finitely generated.

3. Recall that any invertible sheaf $\mathcal{L}$ on a scheme X gives rise to a canonical morphism $W \to \operatorname{Proj}(\Gamma_+(\mathcal{L}))$, where W is an open subset of X, the union of the sets X_f for $f \in \mathcal{L}(X)$, and $\Gamma_+(\mathcal{L})$ is the graded algebra $\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}(X)$.

Let X be the non-separated gluing of two copies of $Y = \mathbb{A}^1_k = \operatorname{Spec} k[x]$ (k a field) along the open set Y_x.

(a) Classify the invertible sheaves $\mathcal{L}$ on X, up to isomorphism. Which ones are generated by their global sections?

(b) For each $\mathcal{L}$ describe explicitly the open set W and the morphism $W \to \operatorname{Proj}(\Gamma_+(\mathcal{L}))$.

4. Show that every *degree-2 hypersurface* $V(f) \in \mathbb{P}^3_{\mathbb{C}}$, where f is a homogeneous quadratic polynomial in 4 variables, is isomorphic to one of the following:
(i) A non-reduced scheme X such that $X_{\text{red}} \cong \mathbb{P}^2_{\mathbb{C}}$,
(ii) A union of two projective planes $\mathbb{P}^2(\mathbb{C})$ intersecting along a line $\mathbb{P}^1(\mathbb{C})$,
(ii) The projective closure of the cone $z^2 = xy$ in $\mathbb{A}^3$, or
(iii) $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.
To what extent does this classification depend on the ground field being the complex numbers?

5. Let $X = \operatorname{Spec}(\mathbb{C}[z^{\pm 1}])$, that is, X is the scheme such that $X_{\text{cl}} = \mathbb{C}^\times$ as a classical variety over $\mathbb{C}$. Setting $Y = X$, let $f\colon X \to Y$ be the morphism given by $z \mapsto z^2$. The morphism $z \to -z$ then generates an action of the cyclic group G of order 2 on X by automorphisms as a scheme over Y.

In the analytic topology on $\mathbb{C}^\times$, X is a principal G bundle over Y, that is, we can cover $Y = \mathbb{C}^\times$ by open sets U such that $f^{-1}(U) \subseteq X$ is isomorphic to $G \times U$ as a topological space (and also as a complex analytic manifold) equipped with an action of G by automorphisms over U.

(a) Show that we can identify G with the underlying set of an affine group scheme over $\mathbb{C}$ in such a way that it acts algebraically on X. (More generally one can do this for any finite group acting by automorphisms of an algebraic variety.)

(b) Show that the action of the group scheme G on the fiber $f^{-1}(y)$ over each closed point of y is isomorphic to the action of G on itself by left multiplication.

(c) Show that X is not a principal G bundle over Y in the Zariski topology.

6. The set of pairs (A, B) of commuting $n \times n$ matrices, over an algebraically closed field k, is an affine algebraic variety X, defined by obvious equations. By an old theorem of Motzkin and Taussky, X is irreducible.

(a) Given the Motzkin-Taussky theorem, find the dimension of X.

(b) Use this to prove that for every $n \times n$ matrix A, the space of matrices that commute with A has dimension at least n (you can do this without using Motzkin-Taussky by considering a suitable irreducible component of X).

7. Let $G(n, k)$ denote the Grassmann variety (over $\mathbb{C}$) of vector subspaces $V \subseteq \mathbb{C}^n$ of dimension $\dim(V) = k$. In class we used the theorem that projective morphisms are proper to prove that the set of pairs $(V, W) \in G(n, k) \times G(n, l)$ such that $\dim(V \cap W) \geq m$ is a closed subvariety X.

Make this more explicit by proving that X is irreducible (assuming that $m \leq k, l \leq n$, so X is non-empty), finding the dimension of X, and finding homogeneous equations in the Plücker coordinates on $G(n, k) \times G(n, l)$ whose zero locus is X. For a real challenge, prove that your equations actually generate the ideal of X in the homogeneous coordinate ring of $G(n, k) \times G(n, l)$.

8. The degree d *Veronese map* $\mathbb{P}^1_k \to \mathbb{P}^d_k$ is given by $(x : y) \mapsto (x^d : x^{d-1}y : \cdots : xy^{d-1} : y^d)$.

(a) Show that the Veronese map is the same as the projective embedding given by the ample line bundle $\mathcal{L} = \mathcal{O}(d)$ on $\mathbb{P}^1_k$ and a basis of its k-module of global sections.

(b) Assuming for simplicity that k is an algebraically closed field, show that the image of the degree 3 Veronese map, considered as a reduced closed subscheme of $\mathbb{P}^3_k$, is given by $V(I)$ for the graded ideal $I = (u^2 - tv, v^2 - uw, tw - uv)$, in coordinates $(t : u : v : w)$ on $\mathbb{P}^3$. If you omit the last equation, taking $J = (u^2 - tv, v^2 - uw)$, how does $V(J)$ differ (if at all) from $V(I)$?

9. Let f be a homogeneous polynomial of degree d in $n + 1$ variables over a field k, so $X = V(f) \subseteq \mathbb{P}^n_k$ is a *degree d hypersurface*. Compute $H^i(X, \mathcal{O}(m))$ for all i and m. (These cohomology groups are finite dimensional vector spaces over k, whose dimensions depend only on n, d, m and i, and not on the specific choice of f.)